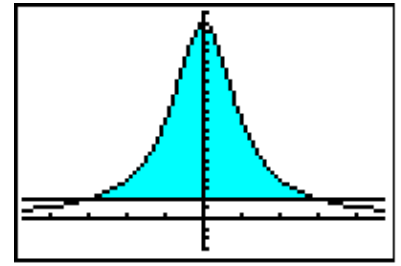
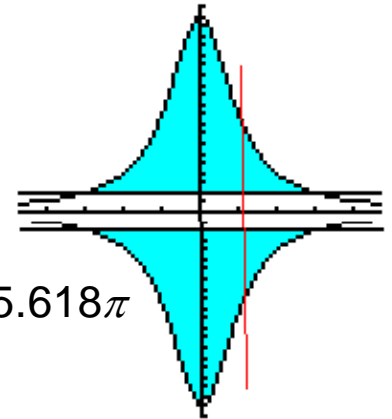


A. Area of R =  $\int_{-3}^3 \left( \frac{20}{1+x^2} - 2 \right) dx = 37.961$  or 37.962

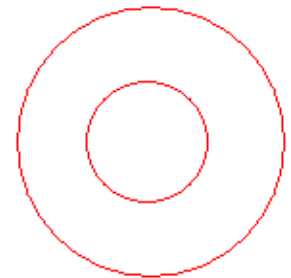


B. Volume =  $\int_{-3}^3 \left( \pi \left( \frac{20}{1+x^2} \right)^2 - \pi 2^2 \right) dx = 1871.190$  or  $595.618\pi$

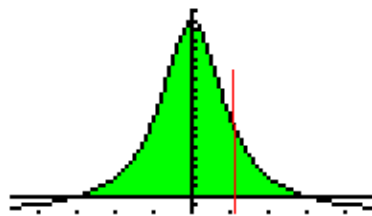


Reflected region for part B

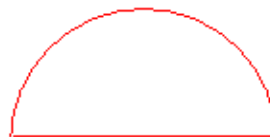
C.  $\int_{-3}^3 \frac{1}{2} \pi \left( \frac{\left( \frac{20}{1+x^2} \right) - 2}{2} \right)^2 dx = 174.268$  or  $55.471\pi$



Typical slice in part B



Base of semicircles in part C



Representative slice in part C. Diameter is the difference of the two functions.

A.

$$\text{Gallons entering tank} = \int_0^7 f(x) dx$$

$$= 8,263.806 \text{ or } 8,263.807$$

or rounded to the nearest gal: 8,264 gallons

B. Net Change in the Amount of water in the tank =

$$A = \begin{cases} \int_0^x (f(t) - g(t)) dt & \text{when } 0 \leq x < 3 \\ \int_0^3 (f(t) - g(t)) dt + \int_3^x (f(t) - g(t)) dt & \text{when } 3 < x \leq 7 \end{cases}$$

so

$$\frac{dA}{dx} = f(x) - g(x) = \begin{cases} f(x) - 250 & \text{when } 0 < x < 3 \\ f(x) - 2000 & \text{when } 3 < x < 7 \end{cases}$$

$$\frac{dA}{dx} = 0 \text{ when } x = 1.617 \text{ or } x = 5.076$$

When  $0 < x < 1.617$  or when  $3 < x < 5.076$ ,
$$\frac{dA}{dx} < 0$$

or the amount of water in the tank is decreasing.

When  $1.617 < x < 3$  or when
$$5.076 < x < 7, \frac{dA}{dx} > 0$$

or the amount of water in the tank is increasing.

Alternatively: Based on the graph of  $f$  and  $g$  we know until  $t = 1.617$  more water is being removed from the tank than is being added. Therefore the amount of water in the tank is decreasing. After  $t = 1.617$  and before  $t = 3$  there is more water being added to the tank therefore the amount of water is increasing. After  $t = 3$  until  $t = 5.076$  there is more water being taken from the tank than is being added. Therefore the amount of water is decreasing. Then from  $t = 5.076$  to  $t = 7$  there is more water being added to the tank than is being removed so the amount of water is increasing.

C. There are two candidates for the location for when the amount of water in the tank is greatest. This would be at the place where the amount of water begins decreasing ( $t = 0$ ) and where the amount of water in the tank changes from increasing to decreasing ( $t = 3$ ) or at the right hand endpoint of an interval where the amount of water in the tank is increasing ( $t = 7$ ).

$$A(0) = 5000$$

$$A(3) = 5000 + \int_0^3 (f(t) - 250) dt = 5126.590 \text{ or } 5127 \text{ gallons}$$

$$A(7) = 5000 + \int_0^3 (f(t) - 250) dt + \int_3^7 (f(t) - 2000) dt = 4514 \text{ gallons}$$

So the greatest amount of water in the tank is at time  $t = 3$  when it is 5127 gallons.

## 2007 AB3

A.  $h(1) = f(g(1)) - 6 = f(2) - 6 = 9 - 6 = 3$   
 $h(3) = f(g(3)) - 6 = f(4) - 6 = -1 - 6 = -7$

Therefore by the IVT since  $h(1) = 3$  and  $h(3) = -7$ , there must be a  $1 < r < 3$  where  $h(r)$  takes on the value of  $-5$ .

B.  $\frac{h(3) - h(1)}{3 - 1} = \frac{-7 - 3}{3 - 1} = \frac{-10}{2} = -5$ . Since the average rate of change of  $h$  between  $1 < x < 3$

is equal to  $-5$  and  $f$  and  $g$  are differentiable then there exists a point  $c$  ( $1 < c < 3$ ) where the slope of  $h$  will also be  $-5$ .

C.  $w'(x) = f(g(x))g'(x)$  so

$$w'(3) = f(g(3))g'(3)$$

$$= f(4) \cdot 2$$

$$= -1 \cdot 2$$

$$= -2$$

D. Need to know  $(g^{-1})'(2) = \frac{1}{g'(1)} = \frac{1}{5}$  and  $g(1) = 2$  so the tangent line is

$$y = \frac{1}{5}(x - 2) + 1 = \frac{1}{5}x + \frac{3}{5}$$

A.  $x(t) = e^{-t} \sin t$   
 $x'(t) = e^{-t} \cos t - e^{-t} \sin t = e^{-t}(\cos t - \sin t)$

$$x(t) = 0 \text{ where } \sin t = \cos t \text{ or } t = \frac{\pi}{4}, \frac{5\pi}{4}$$

$x'(t)$  is negative for  $\frac{\pi}{4} < t < \frac{5\pi}{4}$  and positive for  $0 < t < \frac{\pi}{4}$  or  $\frac{5\pi}{4} < t < 2\pi$ . When  $x'(t)$

is negative then the particle is moving to the left. When  $x'(t)$  is positive the particle is moving to the right. There are two candidates for the location where the particle has reached the farthest to the left: at the start of the problem when the particle begins to move

to the right or when  $t = \frac{5\pi}{4}$  when the particle has stopped going to the left and begins to

move to the right.  $x(0) = 0$  and  $x\left(\frac{5\pi}{4}\right) = -e^{-\frac{5\pi}{4}} \cdot \frac{\sqrt{2}}{2} < 0$ . So it is farthest to the left

at  $t = \frac{5\pi}{4}$ .

$$x''(t) = -e^{-t}(\cos t - \sin t) + e^{-t}(-\sin t - \cos t)$$

B.  $= -e^{-t} \cos t + e^{-t} \sin t - e^{-t} \sin t - e^{-t} \cos t$   
 $= -2e^{-t} \cos t$

so  $Ax''(t) + x'(t) + x(t)$

$$Ax''(t) + x'(t) + x(t) = A(-2e^{-t} \cos t) + e^{-t} \cos t - e^{-t} \sin t + e^{-t} \sin t = 0$$

$$e^{-t} \cos t(-2A + 1) = 0$$

$$A = \frac{1}{2}$$

A.  $r(5)=30$  and  $r'(5)=2$  so the tangent line at  $t = 5$  is  $y=2(x-5)+30$ .

$y(5.4) = 2(5.4-5)+30$  or  $30.8$ .  $r''(x) < 0$  for all  $x$  so the tangent line approximation is greater than the real value.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

B.  $\left. \frac{dV}{dt} \right|_{t=5} = 4\pi(r(5))^2 \cdot 2$   
 $= 4\pi(30)^2 \cdot 2$   
 $7200\pi$  cu. ft/min

C.  $\int_0^{12} r'(t) dt \approx (2)(4) + 3(2) + 2(1.2) + 4(.6) + 1(.5)$   
 $\approx 193$  feet

$\int_0^{12} r'(t) dt$  Is the change in the radius in feet during the time period 0 to 12 minutes.

D. The approximation in part C is an underestimate of the change in the radius since the graph of  $r$  is concave down.

## 2007 AB6

A.

$$f'(x) = \frac{k}{2\sqrt{x}} - \frac{1}{x}$$

$$= \frac{k\sqrt{x} - 2}{2x}$$

and

$$f''(x) = -\frac{k}{4}x^{-3/2} + x^{-2}$$

$$= -\frac{k}{4x^{3/2}} + \frac{1}{x^2} = \frac{-kx^{1/2} + 4}{4x^2}$$

B. A critical point occurs when  $f'(x) = 0$  or is undefined.  $f'(x)$  is never undefined.

$$k\sqrt{x} - 2 = 0$$

$f'(x) = 0$  when

$$k = \frac{2}{\sqrt{x}}$$

So if a critical value occurs when  $x = 1$  then  $k = 2$ . To the left of  $x = 1$   $f'(x) < 0$  (so  $f$  is decreasing) and to the right of  $x = 1$   $f'(x) > 0$  (so  $f$  is increasing). Therefore there is a minimum at  $x = 1$ .

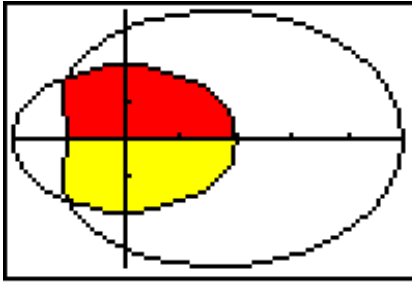
C. The function  $f$  has a zero when  $f(x) = 0$  or  $k\sqrt{x} - \ln x = 0$ . This means  $\ln x = k\sqrt{x}$  or

$k = \frac{\ln x}{\sqrt{x}}$ .  $f''(x)$  changes sign when  $f''(x) = 0$ . This means  $\frac{-kx^{1/2} + 4}{4x^2}$  must change sign.

Combining the two yields  $\frac{-\left(\frac{\ln x}{\sqrt{x}}\right)x^{1/2} + 4}{4x^2}$ . The numerator changes sign when the numerator

equals zero.  $-\left(\frac{\ln x}{\sqrt{x}}\right)x^{1/2} + 4 = 0$  or  $\ln x = 4$  or  $x = e^4$ . So  $k = \frac{\ln e^4}{\sqrt{e^4}} = \frac{4}{e^2}$

## 2007 BC3



A. Using symmetry of the red and yellow area the Area of R =

$$2 \cdot \frac{1}{2} \int_0^{2\pi/3} 2^2 d\theta + 2 \cdot \frac{1}{2} \int_{2\pi/3}^{\pi} (3 + 2\cos\theta)^2 d\theta = 10.370$$

B.

$$\frac{dr}{dt} = \frac{dr}{d\theta} = \frac{d(3 + 2\cos\theta)}{d\theta} = -2\sin\theta \Big|_{\theta=\pi/3} = -2 \cdot \frac{\sqrt{3}}{2} = -\sqrt{3}$$

At the time when  $\theta = \frac{\pi}{3}$  the particle is at a position

$$\left( \left( 3 + 2\cos\frac{\pi}{3} \right) \cos\frac{\pi}{3}, \left( 3 + 2\cos\frac{\pi}{3} \right) \sin\frac{\pi}{3} \right)$$

$$(2, 2\sqrt{3})$$

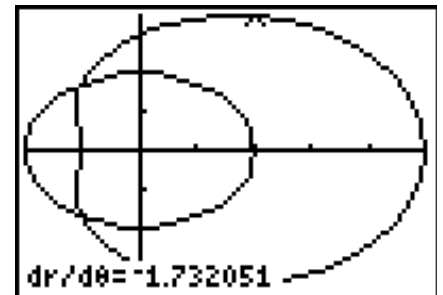
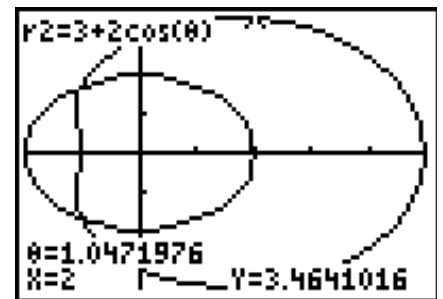
and tracing out the curve in the polar coordinate axes.

$\frac{dr}{d\theta}$  is equal to the rate at which the directed distance from the origin to the point is changing with respect to  $\theta$ . So at this point the distance is decreasing at a rate of  $\sqrt{3}$  units/radian

C

$$\begin{aligned} \frac{dy}{dt} &= \frac{dy}{d\theta} = \frac{d((3 + 2\cos\theta)\sin\theta)}{d\theta} \\ &= -2\sin\theta\sin\theta + \cos\theta(3 + 2\cos\theta) \\ &= -2\sin^2\theta + 3\cos\theta + 2\cos^2\theta \Big|_{\theta=\pi/3} \\ &= -2\sin^2\frac{\pi}{3} + 3\cos\frac{\pi}{3} + 2\cos^2\frac{\pi}{3} \\ &= -2 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 + 3 \cdot \frac{1}{2} + 2 \left(\frac{1}{2}\right)^2 \\ &= -\frac{3}{2} + \frac{3}{2} + \frac{1}{2} = \frac{1}{2} \end{aligned}$$

The particle is located at  $(2, 2\sqrt{3})$  at the instant  $\theta = \frac{\pi}{3}$  and the vertical distance is increasing at the rate of  $\frac{1}{2}$  unit per radian.



## 2007 BC4

A.  $f(e)=2$  and  $f'(e)=e^2 \ln e = e^2$  so the tangent line at  $(e,2)$  is  $y = e^2(x - e) + 2$ .

B.

$$f''(x) = x^2 \cdot \frac{1}{x} + (\ln x)(2x)$$

$$= x + 2x \ln x$$

So for  $1 < x < 3$ ,  $f''(x)$  is positive indicating that  $f$  is concave up.

C.

$$f(x) = \int x^2 \ln x dx$$

$$= (\ln x) \left( \frac{x^3}{3} \right) - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$f(x) = \int x^2 \ln x dx$$

$$= \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx$$

$$u = \ln x \quad dv = x^2 dx$$

$$du = \frac{1}{x} dx \quad v = \frac{x^3}{3} \quad \text{So}$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

$$f(e) = \frac{e^3 \ln e}{3} - \frac{e^3}{9} + c = 2 \Rightarrow c = 2 - \frac{2e^3}{9}$$

$$f(x) = \frac{x^3 \ln x}{3} - \frac{x^3}{9} + 2 - \frac{2e^3}{9}$$

A.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

So

$$\begin{aligned} e^{-x^2} &= 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} \\ &= 1 - (x^2) + \frac{(x^4)}{2!} - \frac{(x^6)}{3!} \end{aligned}$$

B.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 - x^2 - \left( 1 - (x^2) + \frac{(x^2)^2}{2!} - \frac{(x^2)^3}{3!} \right)}{x^4} \\ &= \lim_{x \rightarrow 0} \frac{\left( -\frac{x^4}{2} + \frac{x^6}{6} \right)}{x^4} \\ &= \lim_{x \rightarrow 0} \left( -\frac{1}{2} + \frac{x^2}{6} \right) \\ &= -\frac{1}{2} \end{aligned}$$

C.

$$\begin{aligned} \int_0^x e^{-t^2} dt &= \int_0^x \left( 1 - (t^2) + \frac{(t^2)^2}{2!} - \frac{(t^2)^3}{3!} + \dots \right) dt \\ &= t - \frac{t^3}{3} + \frac{t^5}{10} - \frac{t^7}{42} + \dots \Big|_0^x \\ &= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \dots \end{aligned}$$

$$\begin{aligned} \int_0^{1/2} e^{-t^2} dt &\approx \left( \frac{1}{2} \right) - \frac{\left( \frac{1}{2} \right)^3}{3} \\ &\text{or } \frac{1}{2} - \frac{1}{24} \text{ or } \frac{11}{24} \end{aligned}$$

D. The estimate found in part c differs from the actual value of  $\int_0^{-1/2} e^{-t^2} dt$  by less than  $1/200$  because when the integral is estimated with a certain number of terms of the Taylor Series the error is less than the absolute value of the next term or in this case  $\frac{\left(\frac{1}{2}\right)^5}{10} = \frac{1}{3200}$ . And we know that  $1/3200$  is less than  $1/200$ . So the error is less than  $1/200$ .